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Study of Turbulent Flow with Sensitivity Analysis

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A new type of analysis is introduced that can be used in numerical fluid mechanics. The method is known as sensitivity analysis, and it has been widely used in the field of automatic control theory. Sensitivity analysis addresses in a systematic way the question of "how" the solution to an equation will change due to variations in the equation's parameters and boundary conditions. An important application is turbulent flow where there exists a large uncertainty in the models used for closure. In the present work the analysis is applied to the three-dimensional planetary boundary-layer equations, and sensitivity equations are generated for various parameters in turbulence model. The solution of these equations with the proper techniques leads to considerable insight into the flowfield and its dependence on turbulence parameters. Also, the analysis allows for unique decompositions of the parameter dependence and is efficient.

Introduction

THIS paper is concerned with a relatively new technique that can be used to evaluate the validity of turbulence calculations and models. The technique used is called sensitivity analysis, and it allows an investigator to obtain a measure of sensitivity or solution dependence on the parameters in the turbulence models. As is well known, the present capabilities in turbulent flow modeling depend very strongly on the value of semiempirical or empirical "constants." Many detailed numerical investigations are presently being carried out for flow configurations that are quite far from the conditions of the determination of model empirical coefficients, and there is a definite need to estimate the uncertainty of the solution due to uncertainty in the modeling. Given the present capability to model turbulent flow and ever-growing computational resources, the need to evaluate solution sensitivities will be even greater in the future.

The basic sensitivity analysis technique has been applied for many years in the field of automatic control theory[1-3]; it differs considerably from the usual approach of perturbing a parameter a known amount and evaluating the new results. With sensitivity analysis the model equations are differentiated with respect to the parameters in question and sensitivity equations are thus generated. The solution of these equations can be used to adjust model constants or to develop better physical insight into the model. For example, if an effect in one region of a flowfield does not influence the flow in another region, the sensitivity analysis can be used to verify the fact that the equations used to model an event were physically consistent. In order to illustrate the use of the analysis, a three-dimensional boundary-layer flow with variable density effects and roughness will be calculated and analyzed.

Basic Equations and Analysis

The basic equations used in this work describe the planetary boundary layer.[5,6] These equations with cross flow, but not cross-flow functional variation, are

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}-vf=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) \tag{1}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial v}{\partial z}+uf=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

where u and v are the mean horizontal velocities and w the vertical velocity component. The effects of turbulence, density variation, and wall roughness have been represented by an exchange coefficient K and the Coriolis influences are contained in f.

A great deal of uncertainty exists about the functional form that the exchange coefficient K takes in many flows in the atmosphere. In the present work a rather simple form will be used[6]; however, it will be more than adequate to illustrate sensitivity analysis. It is known from observations that K depends on atmospheric stratification or Richardson number R, the von Kármán constant k_0, the roughness parameter Z_0, and the high altitude mixing length λ. The form used is

$$K=\ell^2\frac{\partial u}{\partial z}(1+\alpha R_i)^{\pm 2} \qquad \begin{matrix} +2\rightarrow R_i<0 \\ -2\rightarrow R_i>0 \end{matrix}$$

$$=k_0(Z+Z_0)\Big/\left[1+\frac{k_0(Z+Z_0)}{\lambda}\right] \tag{4}$$

With this formulation, all of the parameters mentioned above can be grouped into a single parameter, $\beta=\ell^2(1\pm\alpha R_i)^{\pm 2}$, and, as will be shown, all sensitivity results can then be obtained from this single parameter β.

In order to form the sensitivity equations for β from Eqs. (1-3) we differentiate these equations with respect to β and obtain

$$S_u\frac{\partial u}{\partial x}+u\frac{\partial S_u}{\partial x}+S_w\frac{\partial u}{\partial z}+w\frac{\partial S_u}{\partial z}-fS_v=\frac{\partial}{\partial z}K\frac{\partial S_u}{\partial z}+\frac{\partial K}{\partial \beta}\frac{\partial U}{\partial z} \tag{5}$$

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$$S_u \frac{\partial v}{\partial x} + u\frac{\partial S_v}{\partial x} + S_w \frac{\partial v}{\partial z} + w\frac{\partial S_u}{\partial z} - fS_u = \frac{\partial}{\partial z} K\frac{\partial S_v}{\partial z} + \frac{\partial K}{\partial \beta}\frac{\partial v}{\partial z} \tag{6}$$

$$\frac{\partial S_u}{\partial x} + \frac{\partial S_w}{\partial z} = 0 \tag{7}$$

where

$$S_u = \frac{\partial u}{\partial \beta}, \quad S_v = \frac{\partial v}{\partial \beta}, \text{ and } S_w = \frac{\partial w}{\partial \beta}$$

The above equations are linear in the sensitivity variables S_u, S_v, and S_w and retain the parabolic nature of Eqs. (1-3). In most cases, the solution method for the sensitivity equations is identical to that employed for the original transport equations. For both Eqs. (1-3) and (5-7) an implicit finite difference scheme developed by Peterson[7] has been used.

The use of the collective parameter β can now be illustrated. The advantage of employing a parameter β is that the chain rule may be used once the group sensitivity is known. For example, if $\partial u/\partial\beta$ is calculated from Eq. (5), then by using the relationship

$$\frac{\partial u}{\partial k_0} = \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial k_0}$$

the sensitivity of the primary flow velocity to the von Kármán constant can be found without any additional calculations. Therefore, only one calculation is necessary in order to obtain a whole family of sensitivity parameters, and considerable computational time can be eliminated.

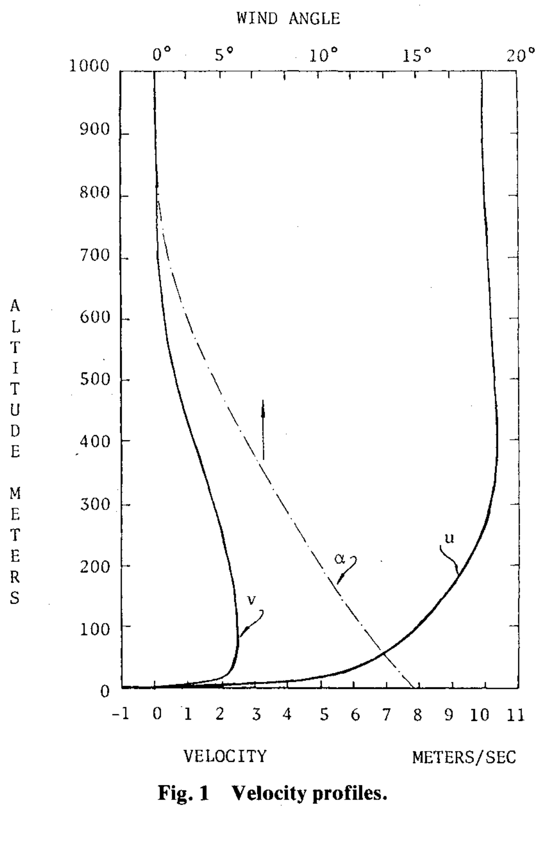

Fig. 1 Velocity profiles.

A valuable extension of the sensitivity analysis is to calculate the sensitivity of the velocity field with respect to the value of a parameter at one spatial location. For example, Eq. (1) could be differentiated with respect to the value of β at $z = 0.4$ and one obtains an equation for

$$S_{u_{z=0.4}} = \frac{\partial u}{\partial \beta}_{z=0.4}$$

with the boundary conditions $S_u(0) = S_u(2) = 0$.

A complete analysis of all of the β parameters at all of the node points in a finite difference simulation will generate a matrix of sensitivities:

$$\hat{\hat{S}} = \begin{matrix} \frac{\partial u_1}{\partial \beta_1} & \frac{\partial u_1}{\partial \beta_2} & \frac{\partial u_1}{\partial \beta_3} & \cdots \\ \frac{\partial u_2}{\partial \beta_1} & & & \\ \frac{\partial u_3}{\partial \beta_1} & & & \\ \vdots & & & \end{matrix}$$

Summing any row of the matrix S will agree with the solution of Eq. (5) at any particular Z station Z_1, while summation of a column will give the same result as the solution to the equation for $S_{u_{Z=0.4}}$ [this equation will have a form very similar to Eq. (5)]. The magnitude of a term $\partial u_i/\partial\beta_j$ will be a function of step size and $\partial u_i/\partial \beta_j \to 0$ as $\Delta Z \to 0$. However, the total contribution does converge to $\partial u_i/\partial\beta$ in this limit. It will be shown in the results section of the paper that the above technique of selective or partial sensitivity is one of the most useful tools in sensitivity analysis.

Another interesting feature of sensitivity analysis is concerned with the presentation of the sensitivity coefficient. For example, the quantity $\partial u/\partial\beta\ \beta/u$ can be directly interpreted as a percentage change in u for a 1% change in the parameter B. A second useful form for presenting sensitivity analysis is called the logarithmic trajectory function[3] defined by $\partial u/\partial \ln\beta = \partial u/\partial\beta\ \beta$. This function gives the magnitude of the change u for a 100% change in β. By using all three representations, a variety of information can be built upon the dependence of the solution on the parameters in the turbulence model. Some typical results will be presented in the next section of the paper.

Results

Sensitivity calculations will now be presented for flow in the planetary boundary layer. Although the turbulence model is not the most advanced, it does reproduce the essential features of the turbulent boundary layer and it illustrates the use of sensitivity analysis in a clear fashion. Also, there is no difference in principle, as far as sensitivity analysis is concerned, between the use of simple or complex turbulence models. The basic flow consists of a 1000-m-thick boundary layer with a mean velocity u of 10 m/s. As the flow approaches the surface the Coriolis terms cause the cross flow to develop and a wind angle of approximately 20 deg develops at the surface[7] (Fig. 1).

Because the sensitivity equations depend on the mean velocity profiles, a variety of roughness heights Z_0 and values of Richardson numbers ($R_i = -0.5$, 0, $+0.5$) were employed. The Richardson numbers of $+0.5$ and -0.5 may seem to be at the stability extremes but these values were selected to exhibit clearly the sensitivities due to stability consideration.

Shown in Figs. 2-4 are the primary flow velocity sensitivities for the von Kármán constant k_0 for the three extremes of R_i. The von Kármán constant determines the

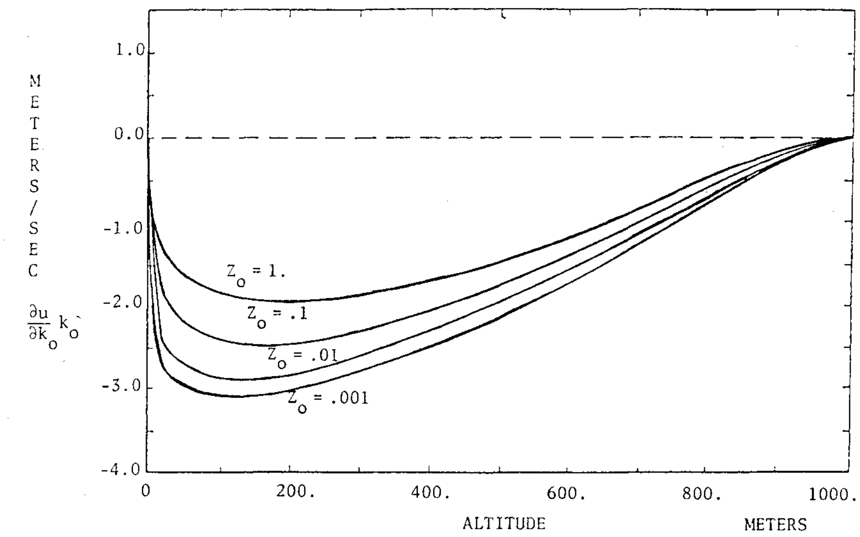

Fig. 2 von Kármán constant sensitivities for $R_i = -0.5$.

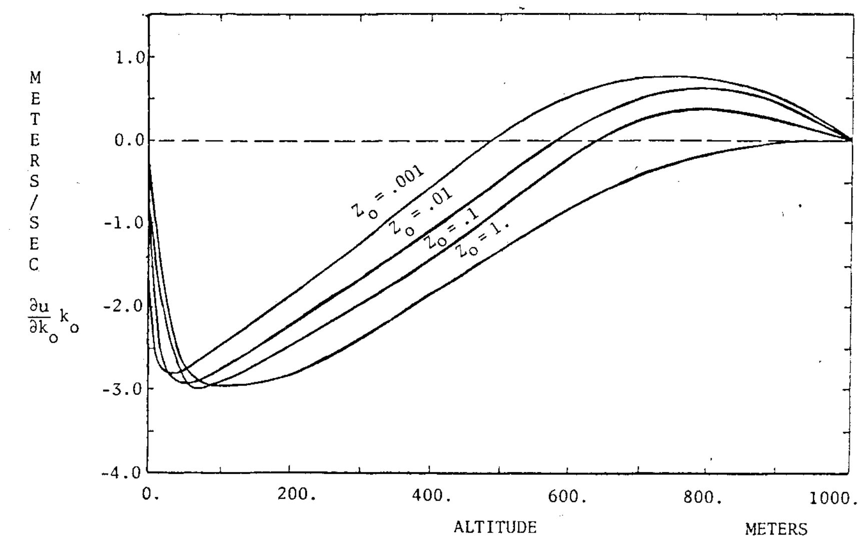

Fig. 3 von Kármán constant sensitivities for $R_i = 0$.

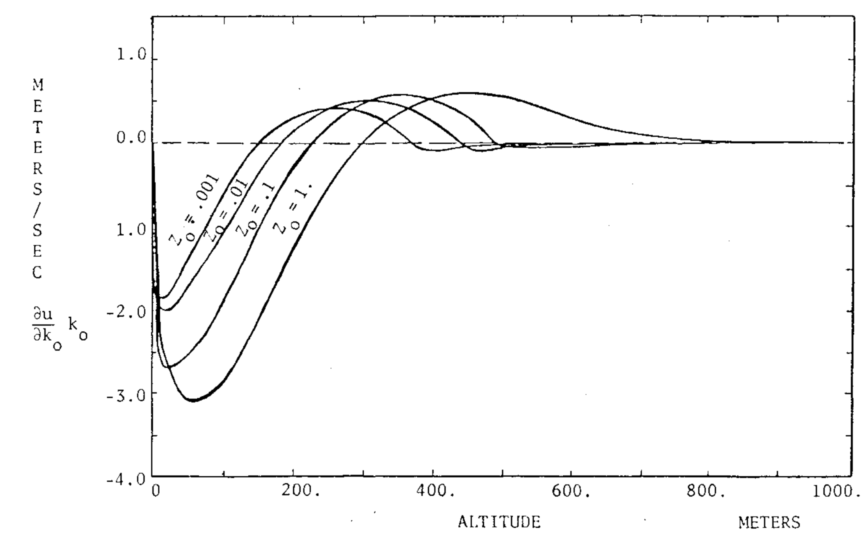

Fig. 4 von Kármán constant sensitivities for $R_i = +0.5$.

growth of the turbulence scale as one moves away from the wall and can be said to be a "key" constant for the near-wall turbulent flow. This strong dependence of the inner flow on k_0 is seen very clearly by the large values of the sensitivities $\partial u/\partial k_0 k_0$ near the wall. Also, it should be noted that the sign and the amplitude of the sensitivity coefficient are similar for all R_i in the near-wall region. However, there are large differences for the dependence in the outer region of the flow for varying Richardson number. This dependence is damaging for the turbulence model used in this paper because k_0 is an inner layer parameter and should not strongly influence the outer layer flow. Also, it should be pointed out that the wall stress sensitivity to k_0 is even greater than u because the sensitivities are large near the wall.

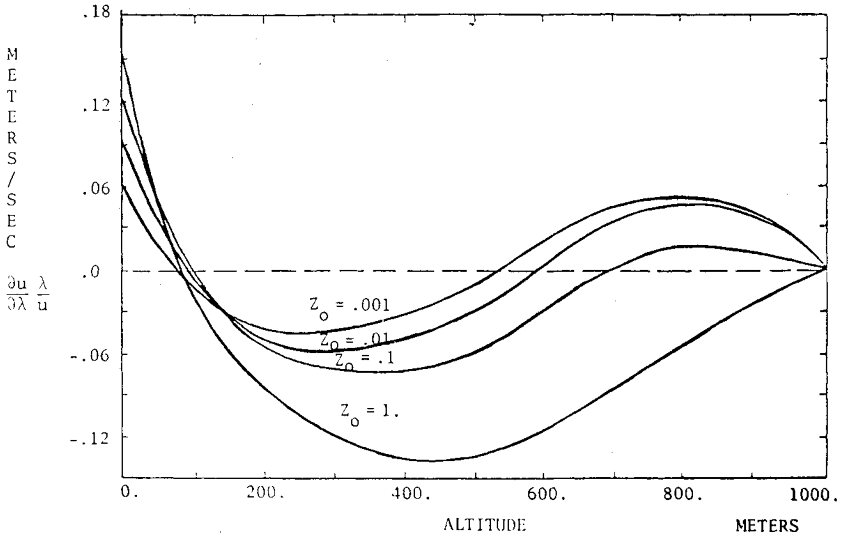

Fig. 5 High altitude mixing coefficient sensitivities for $R_i = 0$.

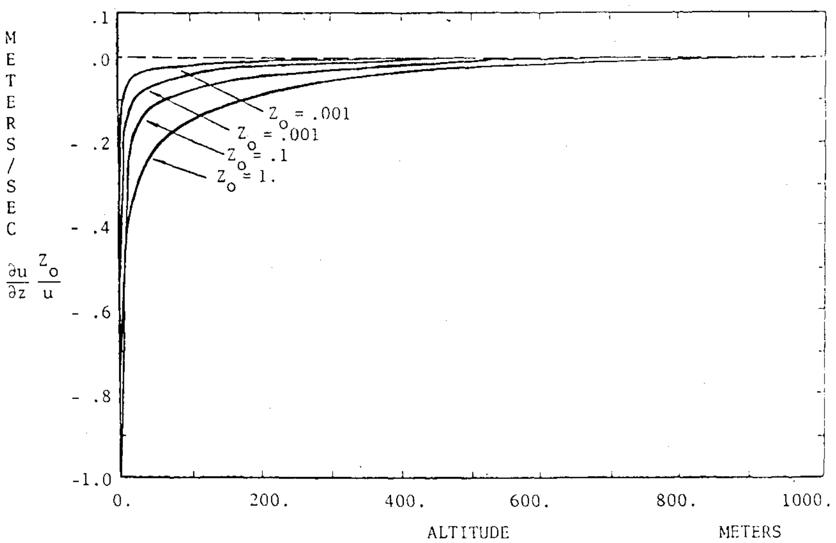

Fig. 6 Velocity sensitivities with wall roughness $R_i = 0$.

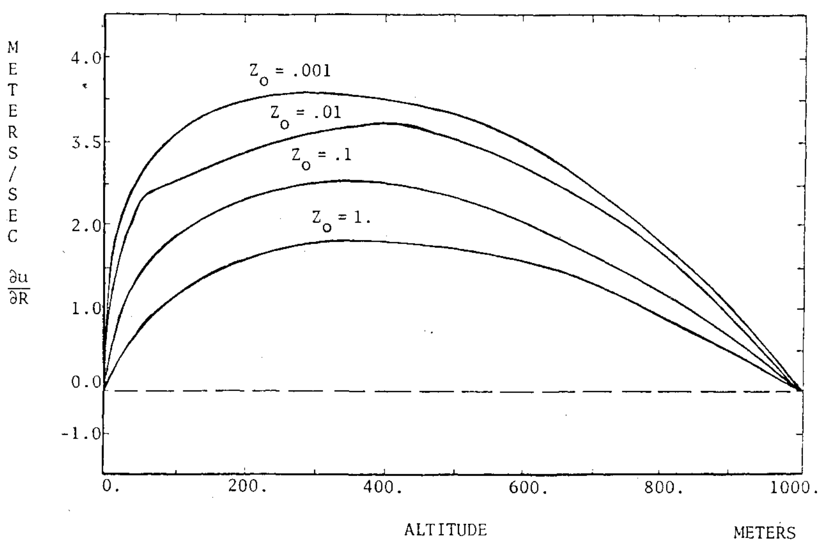

Fig. 7 Velocity sensitivities with $R_i = -0.5$.

A study of one of the parameters in the outer part of the flow is shown in Fig. 5, where the sensitivity to the high-altitude mixing constant λ is shown. The purpose of this parameter is to scale the turbulence in the outer region of the flow. It can be seen from Fig. 5 that there is a strong dependence of the outer flow on λ, but that the inner flow is also sensitive, especially with the wall roughness Z_0. The interaction with Z_0 is due to the functional form used to introduce roughness, and it is doubtful whether this effect agrees with the physics.

The dependence of the flow on Z_0 seems to be properly represented, as can be seen from Fig. 6. Increases of wall

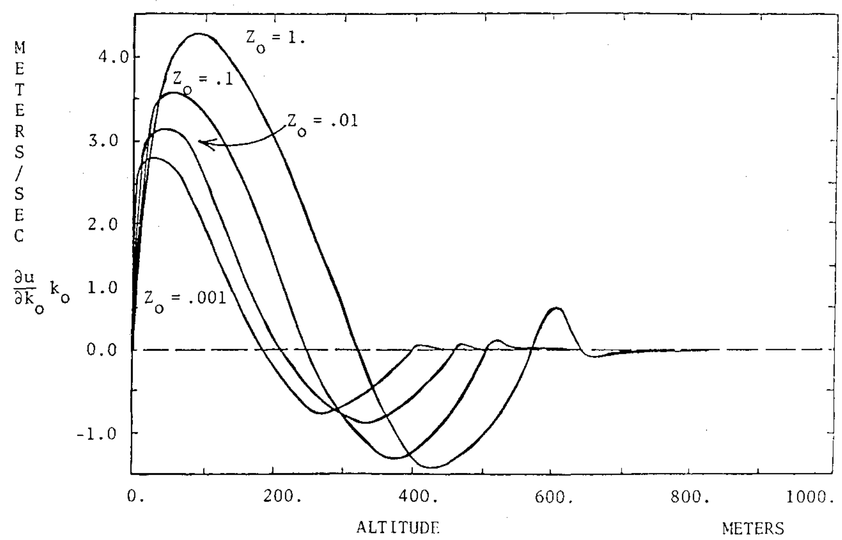

Fig. 8 Velocity sensitivities with $R_i = +0.5$.

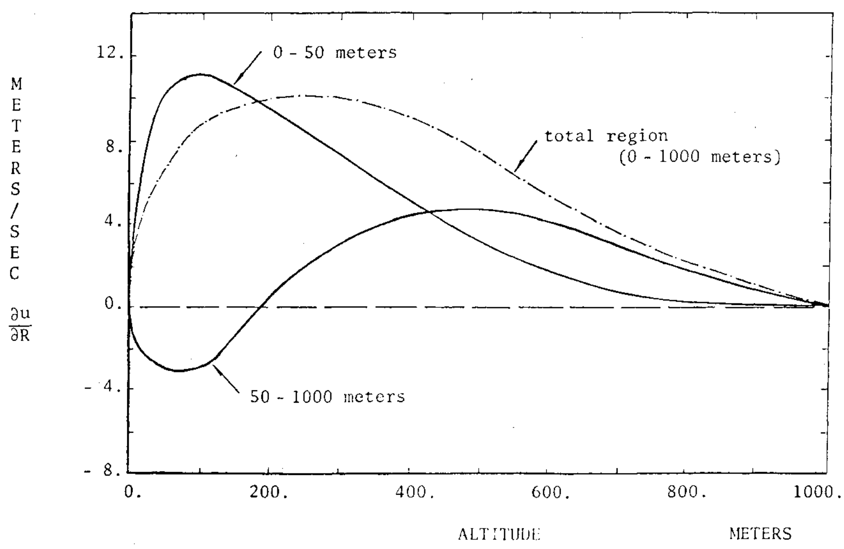

Fig. 9 Spatially selective sensitivities with $R_i = 0$, $Z_0 = 1$.

roughness only cause changes near the wall; the outer flow is not influenced greatly. This ability of a parameter to influence only the region of flow for which it was designed should be an important criterion in selecting models.

The dependence of the flow on the Richardson number is shown in Figs. 7 and 8. Both figures indicate that the sensitivity coefficients are positive near the wall and that the new wall velocity will increase for increasing R_i. This change appears to be in the wrong direction, and physical observations have shown that the wind velocities decrease when approaching a more stable condition (R_i increasing). This problem is not as severe as it first appears because the wall shear does give the correct dependence on R_i. The correct dependence of wall shear on R_i is due to the explicit dependence of R_i in the expression for τ, but there is no explicit dependence of u on R_i. Therefore, a correct result for τ has resulted in an incorrect velocity field.

The problem with R_i can be further studied with the use of spatially selective sensitivity analysis. In order to understand the contribution of the inner and outer layers of the flow two separate sensitivity analyses were performed. In the first analysis R_i was allowed to vary in the first 50 m of the boundary layer, while in the second calculation it varied only in the outer region 50 to 1000 m. The results are shown in Fig. 9, along with the total sensitivity coefficient. As can be seen from the graphs, the positive sensitivity coefficient is due mainly to the variation of the Richardson number in the first 50 m of the atmosphere. Since it is known[7] that this region is essentially neutral, it can be seen that the problem could be removed by having a variable R_i ($R_i = 0$ in the near-wall region). Thus, it has been shown how sensitivity analysis can provide physical insight and suggest possible improvement in turbulent models.

Conclusions

The present paper has exhibited how sensitivity analysis can be used to understand modeling constants in turbulent flow. At the present time there is a need for this type of analysis because of the large uncertainties that are associated with closure constants used in most turbulence models. These constants have been obtained from experimental results on relatively simple flows, and the experimental job of verification of new constants for more complex flows is difficult and expensive. Sensitivity analysis applied to a given model will indicate what regions and parameters in a flow should be focused on, and thus allow for a more efficient use of experimental resources. For the atmospheric boundary-layer problem studied, the analysis showed that the flow was most sensitive to the Richardson number. The technique also identified a problem with the exchange coefficient law under nonneutral conditions, and indicated how the law could possibly be modified to give improved results.

The methods that have been developed in this paper should be generally applicable to other areas in fluid mechanics. The analysis technique depends on having the solution to the fluid equations and being able to solve the sensitivity equations. Also, the results predicted are limited to small changes in the parameter under question. However, the detailed information given, both local and global, is of tremendous value in understanding the relationship between a parameter and the solution to an equation.

Acknowledgment

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